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MATHEMATICS: THE SUBJECT AND THE TEACHER.*

By Frederick C. Ferry.

These are days of change in educational affairs. Greek has almost disappeared from the curricula of school and college. It has entirely disappeared from the individual schedules of all save a mere handful of our students. The contest for the survival of Latin is being hard fought and some of its champions prophesy early failure. Very few men's colleges in New England any longer demand for admission to the Arts course the traditional four years of Latin. The Bachelor of Science course without any Latin at all has become the common thing. We in the far east will hardly be allowed by ourselves to decide the question of the survival of Latin; the great state universities of the middle west promise to play the chief part in the solution of such educational problems, and we probably shall feel obliged to accept their verdicts. Verily "the old order changeth, yielding place to new." Is one to see all the old-fashioned subjects drop out in their turn, and will mathematics soon depart forever from the student's schedule? It requires no excess of confidence to answer no to this question. One notices that the onslaught of the progressives with their weapons of introductory social science, introductory general science, practical arts, community civics, typewriting, forging, and domestic science is aimed at the foreign languages, particularly Latin, and not at mathematics. A careful examination of several of those new curricula which promise to make the high school course really useful, and at the same time easy and delightful to the pupil, shows that they retain in every instance as required subjects for all students algebra and geometry. Naturally these two branches of mathematics cannot be expected to add to the ease or largely to the delight of the average boy or girl's high school course; they are distasteful to many an earnest but non-mathematical student; they have no charm of educational novelty; yet there they stand as subjects required of all secondary school boys and girls, and why? Because they are known to be in many small ways useful subjects and in a large way necessary subjects. As to the future, one may feel sure that mathematics, which began with the first ordering of things, will not pass out from its place in our educational life until such time as order in the universe shall have given way to chaos. For mathematics, as Professor Story says, occupies a peculiar position with respect to both the arts and the sciences. It is, *par excellence,* an art, inasmuch as its chief function is to solve problems—not textbook examples which are merely exercises in the application of methods—but all those problems in human experience for whose correct solution sufficient data are at hand. When any line of investigation, to whatever subject it may refer, has been carried so far as to permit the application of exact reasoning, mathematics assumes control. It determines whether the results presented are consistent and whether the conclusions are valid, and it is mathematics that provides the means for applying these conclusions to the prediction of phenomena not yet observed. No science and no branch of technology is exact unless it rests on a mathematical foundation. The secrets of the future can be foretold for any science only in so far as that science is obedient to the laws of mathematics. Astronomy long ago had accumulated enough data of observation to allow its case to be submitted to the mathematician and accordingly astronomy began long ago to be an exact science. It was the mathematicians, Adams and Leverrier, who told the astronomers that their data called for another planet, and indicated to them the direction in which the astronomer should point his telescope to find it. After astronomy, next came physics, and then applied mechanics, in both of which a sound mathematical foundation has long been in building. Other sciences are in the inductive stage, still collecting material to be put in order by the mathematician as soon as the material is sufficient for the purpose. The calculus is already being applied to some of the problems of economics. The chemist employs mathematical formulas daily. The geologist, the physiologist, and the psychologist all use mathematical tools. And mathematics must continue to be used for such tasks until all science shall have become perfect and shall have passed away forever. Mathematics, then, is one of those few themes which, in Kipling's phrase,

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> " Shall be sung through planets young
> When this is clean destroyed."

But mathematics is not only an art, weighing and arranging and interpreting the data of the sciences. It is also a science, since it has accumulated a vast store of systematic knowledge involving and leading to the methods that it uses in its solutions. These methods are of a very peculiar nature. They differ so widely from all other methods that a special course of training is necessary to him who would use them. And these methods are so many and so varied that no scholar lives long enough to become familiar with them all. Furthermore, new problems are continually arising of such sort as demand new methods; hence mathematics needs the continued service of able minds. Even if the solution of the quadratic equation and all involved therein has been completely determined, the same is probably untrue of the equations of degree three and surely untrue in the case of equations of all higher degrees. Since one may add new variables in countless number to the equation of any degree and may extend the degree in each instance indefinitely, the labor still awaiting the mathematician surpasses the reach of the imagination. It will hardly have been begun when Kipling's "planets young" shall in their turn be "clean destroyed." Mathematics is, then, an art and a science; and its task in either capacity and in both will endure from "everlasting to everlasting."

This estimate of the subject of mathematics is not the one commonly held by the uninitiated. The popular view of mathematics still associates something queer with its name and with those who are its devotees. No doubt it is true that unsound dabblers in the use of mathematical symbols have contributed more than their share to that little collection of books classified in a university library as "Insane Literature." It is there that you will find the "Life Romance of the Algebraist," a remarkable work, which mingles mathematical symbols with sentimental phrases and attempts to set the combination to music; its mathematics is unsound, its verses are unpoetical, and its music is unharmonious. The author recites in the strange medley the hopelessness of his attempts to accomplish two objects,—namely, to win a young lady's affection, and to solve some difficult equations; a rival forces him to relinquish his first object, but no one robs him of his equations, and he continues his vain struggle with them to the end of the crazy volume.

But mathematics has to bear the devotion of the ignorant even more persistently than that of the unbalanced. The problem which best illustrates this is the famous one of "squaring the circle." This demands a method for the construction of a square which shall be exactly equivalent in length of perimeter or area to the circumference or area of a given circle. Thus it involves the ratio of the circumference of a circle to its diameter. It is an ancient problem. One reads in the seventh chapter of the first book of Kings that Hiram of Tyre found this ratio in the case of a molten sea "round all about" to be as "three to one." Aristophanes in the "Birds" introduces a geometer who announces his intention to "square the circle," while Pope, in the "Dunciad," declares the person "mad" who attempts a solution. Archimedes, by a method declared "more divine than human," obtained a close approximation to the 3.14159 which we use. Ptolemy carried the approximation of the famous π still further, while Van Ceulen gave much of his life to its solution and directed just before his death in 1610 that his best result of 35 places of decimals be engraved on his tombstone in St. Peter's Church, Leyden. In our own era Mr. Shanks has extended the result to 707 places,—a line of figures nearly six feet long when set in ordinary type,—and constituting an approximation sufficiently close for any purpose. About twenty-five years ago a German mathematician, Lindemann, established forever that the value of this ratio cannot be given in exact terms. No two numbers whatsoever can represent exactly the ratio of the circumference to the diameter of the circle. Accordingly the circle cannot be squared. But this demonstration has not caused much diminution in the number of so-called solutions printed and distributed throughout the world. Hiram of Tyre was not the first to wrestle with this problem, and the world has no prospect of seeing the last. For some centuries it was supposed that a great reward had been offered by the Queen of England, the French

Academy, or some other distinguished organization for the first correct solution. The great, as well as the ignorant, labored long over it. Some by prayer and supplication sought to reach the desired goal; many a priest in the middle ages reported that he found the problem most highly conducive to his spiritual welfare, and the French mathematician, DeVausenville, announced in 1771 a solution obtained by the direct aid of the Holy Spirit. Several years after Lindemann had proved the problem insoluble, a writer in the *New York Tribune* announced that he had discovered the long-lost secret, the "Nicomedean line," and had established the desired ratio as exactly 32 to 10. It is reported that the legislature of a western state undertook to assist the working mathematician and dispose of the problem forever by fixing the ratio by legislative enactment at 3 to 1. Whether the circumference of the circle was to be forever contracted or the diameter was to be permanently stretched; whether all geometric laws were to be restated and all structures built on mathematical principles were to be shaken from their foundations by the passing of a majority vote in the capitol of a western state, one cannot tell, for the legislature finally decided to leave geometry as it is until, perhaps, the combined attack of several legislatures shake Euclid from his grave and force him to rewrite his famous books. The search for the "Fountain of Youth,"—at least among the native springs of the earth,—has become unfashionable; the chemists and geologists have made us despair of the discovery of the touchstone that shall give us transmutation of metals; electricity is accepted as the nearest approach to the long-sought "perpetual motion"; but the squaring of the circle will still command its earnest laborers. The pseudo-mathematicians will continue to burn their midnight oil over this question and will send through the mail their illogical results. Still the newspapers will publish the solutions of mathematical problems long since proved insoluble. Learning can annihilate learning, but learning cannot annihilate ignorance. "A sword may cut through an iron bar, and the severed ends will not reunite; let it go through the air, and the yielding substance is whole again in a moment."

Attention has been called to the fact that mathematics receives an undue amount of devotion, first, from the unbalanced, and, second, from the ignorant. From both these sources it is enveloped in an atmosphere of which it would perhaps gladly be rid. In the third place, the successful mathematician in many an instance allows the peculiarities of his own personal conduct to color the subject in the eyes of his associates. In particular, the professor of mathematics comes to be regarded as a most absent-minded fellow, whose thoughts linger in vast spaces of his beloved geometry when needed to direct his present movements. So it happens that the professor of mathematics lets his watch get two hours slow and forgets that it is not on time; he leaves a notice in his office window reading, "Will return at two o'clock," and hesitates to enter when he comes back at one-thirty but sits on the steps until he can be sure of arriving at the appointed moment. He forgets that he has declined an invitation to dinner at a friend's house because of another engagement which did not exist and wanders in aimlessness and complete forgetfulness to that very house on the evening of the dinner and lingers in street clothes in the drawing room until the arrival of the well-dressed diners reminds him to run away. It is a professor of mathematics who finally yields to his wife's entreaties to come home and accompany her on an evening's programme of social events; she rejoices at nightfall to see him mindful of his promise and tells him to hurry upstairs and dress so that they may be off; after patiently waiting long in vain for his reappearance, she seeks him in his chamber and finds that he has undressed and gone soundly to sleep in his bed. Numberless are these stories of the professor of mathematics and the most unpleasant thing about them to the mathematician is that they are all true. Probably the mind trained to deal directly with large questions in the field of abstract reasoning is quickest to lose its temporary grasp of the present and the visible; the journeying often in the great spaces of analytical geometry tends to make one absent-minded. Still waters run deep, and the mind engaged in tracing out the singularities of twisted curves in n-dimensional realms may fail to be stirred by the sound of the dinner bell. But one need not hesitate to enter the field of the mathematician's toil because of such facts as these, for one has suffered little from the effects of his profession when these are its worst results.

It would be a more serious reflection on mathematics if all of those unkindly criticisms which some of the mighty are fond of making with regard to the teachers of that subject were deserved. A university president used to like to refer to those "least human of teachers, the mathematicians." It is granted that few teachers of mathematics are human to the extent of marking examination papers without reading them, or even of accepting the inaccurate statement as, by good intention, of the same value as the accurate one. The mathematician finds his subject an argument for precision and he aims at justice with full hope of reaching his goal. Let him not be blamed that he is keen at detecting errors and quick at calling mistakes by their proper names. In such direction has his subject trained him. Again, a university president stated to a little company of college officers that the teacher of mathematics was incapable of forming *moral* judgments. One of his hearers asked him how he would reconcile this statement with the fact that the deans of our colleges and universities are more largely taken from the professors of mathematics than from any other class of college teachers, and that these deans handle the questions of discipline which call constantly for the forming of moral judgments. He replied that he was not aware that the mathematicians were providing more than their share of these officers. In a truth-seeking spirit he at once investigated the matter, found that he was in error, and generously acknowledged the truth to the mathematicians among his associates. That a teacher whose subject demands clear reasoning and insistence on the reaching of definite, irrevocable conclusions should be found capable of dealing with cases of infraction of college law need surprise no one. The habit of reaching definite results and holding them fast is a good habit and such an attitude carries conviction with it. A boy gains confidence in the justice of his punishment if he finds that his sentence is held by the disciplinarian,—not angrily,—but ever so clearly as a deserved thing and ever so firmly as a necessary thing. For the development of this attitude, no better training is found than that afforded by mathematics.

But the text of this rambling discourse on mathematics includes the teacher as well as the subject. He is a strange and unventuresome critic in these days who cannot insist that the teaching of our schools is uninteresting, uninspiring, inefficient, unpedagogical,—a waste of money,—and that the teaching of our colleges is still worse. With a confidence often born of inexperience, the reformer sets forth the faults in each layer of the whole structure of our educational system. What is easier to establish than the proposition that the high school teacher fails to make brilliant scholars of all the unwilling, happy-go-lucky youths who throng the classroom? or that the college teacher cannot make either a scholar or an athlete of many a youth who, for avowed purposes social, or athletic, but not educational, is sent by fond parents for four years of indulgent life in the college atmosphere? That the results are not nearly so bad as they are painted, all who teach are well aware. That, in view of the commercial spirit of the time and place, the lack of intellectual stimulus in the home, and the failure on the part of the public to appreciate what of good is done, the teachers are are doing surprisingly well is a statement void of exaggeration.

Mathematics fares no better and no worse, perhaps, at the hands of critics than the other school subjects. Of it, one reads that it "has outlived its usefulness as a subject of secondary school instruction." "It is too remote from life to interest the student." "There is no such thing as mental discipline, hence mathematical teaching has no value." Such sentences in almost countless number are thickly scattered about in the journals of these days. Still, as Kant said: "A science is exact only in so far as it employs mathematics." "Our entire present civilization," says another, "as far as it depends upon the intellectual penetration and utilization of nature, has its real foundation in the mathematical sciences." Engineering, architecture, navigation, railroad building, and surveying all rest on a mathematical foundation. "There is no royal road to geometry," said Euclid; and when will our boys equip themselves with the necessary mathematics for their parts in our American life unless their teachers give them a good start in this great subject in their early days? Moving pictures are more entertaining for the moment than equations, but to entertain is not a teacher's chief function, the critics to the contrary notwithstanding. Not "what does Peter, the boy, wish now to study?" but "what will Peter, the man, wish twenty or thirty years hence that he had studied?" is the vital question. If he or she is never to come in contact with any development of science which will demand mathematical insight, it still is true that the secondary school curriculum has no other subject nearly so well adapted to the exercise of the reasoning power as geometry. Other subjects may help to a larger acquisition of facts, but the study of mathematics excels in the development of power.

Those who teach freshman mathematics in college feel that the student is too insecurely grounded in the simpler and more commonly used principles, particularly in algebra. As a freshman recently explained, he had never been well grounded in the "fundaments" of the subject. Truly, if all these boys had learned and could not forget the "fundaments," the college teacher of mathematics would have a far happier career. The average freshman knows that he can solve for x the equation $2x = 1$ and immediately, with unwavering confidence, he gives the result $x = 2$. Similarly, the equation $x/2 = 1$ establishes to his full satisfaction that $x = \frac{1}{2}$. The chief need of the student for success in college algebra is the ability to solve quadratics; yet very few of them are at all trustworthy for the performance of that feat unless the equation is easily factorable at sight. One wonders whether it is not possible in school to drill these boys so long and so thoroughly in the solving of equations of the first and second degrees that they cannot forget, even in a long summer vacation. Yet when the college teacher meets the same boys in class in sophomore year and finds that his own systematic drilling of six months ago has gone to "that bourne whence no pilgrim returns" he thinks far less unsympathetically of the high school teacher. To repeat the instruction on the fundamental things again and again and yet again must be the mathematics teacher's task.

The department of mathematics at Williams has striven earnestly to improve its methods of teaching and has experimented with small changes in method from time to time. Some of these slight departures from the ways of teaching which were found there twenty years ago have now been thoroughly tried and permanently adopted. The student has need of much practice in the application of principles and theorems; to secure this practice he is required first to present in writing at the beginning of every recitation, as an imperative part of that day's exercise, from six to twelve appropriate examples, chosen ordinarily from the textbook, and assigned at the previous recitation. The problems are carefully examined by the instructor and an accurate report of the results is posted on the bulletin board before the next exercise. Unless the student presents during the semester at least seventy-five per cent. of these assigned problems correctly solved, he is declared "deficient" at the end of the term and must repeat the course. This requirement is carried out to the letter and the student follows the posted record with great interest. To prevent the postponement of this part of the work and the consequent lagging of the student in the course it is required that at least half the assigned problems be presented in correct form by every student the very day when they are due. Furthermore, the few stray problems in which the student has failed may not be made up at all later than one week after they were assigned. The papers of problems thus submitted are not returned to the student, for it is counted undesirable to have sets of solved problems left permanently where a succeeding class might find them. The different divisions of the class have different assignments,—an arrangement which discourages copying. This laboratory feature of the work has been regularly employed in all the courses for at least a half dozen years and would not willingly be relinquished by any of the five teachers of mathematics. On the pupil's side, it has not at any time aroused much opposition, for the student finds that it is not without value to him.

Again, the department makes sure that no division should at any time include more students than can be accommodated, all together at once, at the blackboards in the recitation rooms which the department uses. Accordingly, a division seldom includes more than fifteen students, and the average number at the present time is less than thirteen. A recitation hour seldom passes without the instructor's sending the entire class to the blackboard to demonstrate theorems and solve problems there. To make sure that the students shall work independently, they are not allowed to take their books to the blackboard and are not permitted to confer with each other there, but are directed to call on the instructor for all needed help. Furthermore, each student has his own individual assignment at the blackboard. To make this possible and to secure that each student has a full and correct statement of his problems before him, the instructors prepare sets of cards for successive recitations. Each card bears problems suited to the lesson of the day; no two cards are alike and the problems for each exercise are either made up by the instructor or selected from other textbooks than that used by the class. Naturally the problems are as varied and call for as much original ability as the instructor can safely demand. To distribute these cards to the class consumes but a moment of time, and each student finds himself at the blackboard face to face with a problem which he alone is to solve. The instructor has the answers on another card in his hand and goes from student to student checking the work, distributing fresh cards, pointing out errors, bestowing help,—actually teaching each individual boy as that particular one shows need. All students take kindly and earnestly to this method. Practically never is one of the cards lost and the sets in use now have remained unbroken in many instances through a half dozen years. None of the staff of teachers regards any other feature of the work of so great profit as this,—with the sets of cards, the large blackboards, and the small divisions.

Each student is called on for oral recitation nearly every day. With these three means of contact, the problems done in the study and presented on paper, the work done on the blackboard, and the oral recitations, the instructor knows what progress his students are making, what principles need further emphasis, and whether his rate of progress is adapted to his class. Teaching keeps its charm under such circumstances. It can hardly do so when the classes are far too large and the possibility of instructing the individual according to his needs is lost.

In some universities, the teaching of all the classes in mathematics is done by lectures. At Williams, the work of nine semester courses rests solidly on textbook foundations, while only a single senior course is taught by lectures. At the same time, in all the courses the more difficult theorems of the next lesson are explained somewhat in advance. This saves the student's time, for it permits him to undertake the study of the theorem in the textbook with some knowledge of the reasoning employed and the goal sought.

From various sources comes to every teacher the demand that his subject be made interesting and show contact with real life. How far can the teacher of mathematics legitimately satisfy this demand? In each branch of mathematics some ingenuity may well be expended in selecting such problems as men actually meet in every-day affairs rather than those artificial ones which never have existed outside an old-fashioned textbook. The field of "industrial mathematics" so-called offers many suggestions in this direction for the use of the classes in algebra and trigonometry. In geometry, one may rouse much interest by leading a student to the consideration of spaces of zero, one, two, three, and even four and higher dimensions. One point determines a point in one-dimensional space; two points determine a line in two-dimensional space; three points determine a plane in three-dimensional space; what, then, by analogy, must four points determine in a four-dimensional space and five points in a five-dimensional space? The four bounding lines of a square, a two-dimensional figure, may be folded down into a one-dimensional space; so may the six bounding planes of the cube be folded down into two-dimensional space; what results if the eight bounding cubes of the corresponding 4-dimensional figure are folded down into 3-dimensional space? Time, as a one-dimensional space, always interests a class of freshmen. An inhabited two-dimensional space, as described in "Flatland, by a Square," with the resultant "Theory of Ghosts," when presented to a college class, proves a lively subject of discussion at many a student dining-table. Mathematics is not lacking in material to interest the student.

After all, the great lesson to be gained by the average student from the study of mathematics is accuracy. That the slightest departure from the straight and narrow path makes the result inaccurate and therefore wrong,—not almost right, but everlastingly wrong,—this is the great lesson for our students. In nearly all other subjects the "nearly right" may answer. The student is accordingly accustomed to a state of well-intentioned uncertainty, which impedes his progress and weakens his character. The teacher of mathematics must impress his pupil's mind indelibly with the fact that the exact alone is acceptable, while the inexact is a thousand times not acceptable. It is a good lot in life to be called to teach the young this lesson. And there is much satisfaction in dealing day by day with a subject where wrong is wrong and right is right forevermore.

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